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THE BORSUK-ULAM THEOREM AND FORMAL GROUP LAWS

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Introduction

The present paper is concerned with the following question raised on the classical Borsuk-Ulam theorem : Let G denote a cyclic group of odd order q, and let Σ be a homotopy $(2n+1)$-sphere on which a free differentiable G-action is given. For any differentiable m-manifold M and any continuous map $f\colon \Sigma\to M$, put $A(f)=\{x\in\Sigma \mid f(x)=f(xg) \text{ for all } g\in G\}$. What can be deduced about the covering dimension of $A(f)$?

In response to this question, the authors showed previously that if q is a prime p then $\dim A(f)\geqq 2n+1-(p-1)m$ ([4], [6]). Furthermore, one of the authors showed in [5] that if q is a prime power p^a and M is the Euclidean space $\boldsymbol{R}^m$ then

$$\begin{aligned}(0.1)\qquad \dim A(f)\geqq &(2n+1)-(p^a-1)m\\ &-[m(a-1)p^a-(ma+2)p^{a-1}+m+3].\end{aligned}$$

It will be shown in this paper that (0.1) still holds for any differentiable m-manifold M.

The procedure taken in this paper is different from the previous ones, and we shall derive the above result from a general theorem stated in connection with the formal group law for some general cohomology theory.

Assume that there is given a multiplicative cohomology theory h defined on the category of finite CW pairs and satisfying the conditions: i) each complex vector bundle is h-orientable, ii) $h^i(pt)=0$ for each odd i. Let $F(x,y)\in h(pt)[[x,y]]$ denote the formal group law associated to h, and $[i](x)\in h(pt)[[x]]$ denote the operation of "multiplication by i" for a positive integer i. We shall show that

$$(0.2)\qquad \dim A(f)<2d \quad\Rightarrow\quad x^d\Big(\prod_{i=1}^{(q-1)/2}[i](x)\Big)^m\in(x^{n+1},\ [q](x)) \quad \text{in } h(pt)[[x]],$$

where (a,b) denotes the ideal generated by a and b.

Take as h the general cohomology theory defined from K-theory. Then it is seen by using elementary algebraic number theory that (0.2) is equivalent to (0.1).

We can also take as h the complex cobordism theory U^*. Since U^* is stronger than K-theory in general, it is expected that sharper result than (0.1) will be obtained from (0.2) applied to $h=U^*$. However we have no method to derive numerical conditions equivalent to (0.2) for $h=U^*$.

In an appendix, we shall prove in the same procedure as above a non-existence theorem for equivariant maps which generalizes the result of Vick [10].

1. The formal group law for a multiplicative cohomology

We recall first some facts on multiplicative cohomology theory (see Dold [3]).

We fix once and for all a multiplicative reduced cohomology theory $\tilde{h}$ defined on the category of finite CW complexes with base point. There is the corresponding multiplicative cohomology theory h defined on the category of finite CW pairs.

Let ξ be a real n-dimensional vector bundle over a finite CW complex B, and denote by $M(\xi)$ the Thom space for ξ. For each $b \in B$ let ξ_b denote the restriction of ξ over b. Then $\tilde{h}(M(\xi_b))$ is a free $h(pt)$-module on one generator. ξ is said to be *h-orientable* if there exists $t(\xi) \in \tilde{h}^n(M(\xi))$ such that $t(\xi)|M(\xi_b)$ is a generator of $\tilde{h}\ (M(\xi_b))$ for each $b \in B$. Such $t(\xi)$ is called an *h-orientation* or a *Thom class* of ξ By an *h-oriented* vector bundle we mean a vector bundle in which an h-orientation is given.

Let $D(\xi)$ (or $S(\xi)$) denote the total space of the disc bundle (or the sphere bundle) associated to ξ, and consider the homomorphism

$$\tilde{h}^n(M(\xi)) = h^n(D(\xi),\ S(\xi)) \xrightarrow{j^*} h^n(D(\xi)) \underset{\cong}{\xrightarrow{p^{*-1}}} h^n(B),$$

where j is the inclusion and p is the projection. The image of $t(\xi)$ under this homomorphism is called the *Euler class* of the h-oriented bundle ξ, and is denoted by $e(\xi)$.

The following facts are easily proved:

(1.1) If there is a bundle map $f\colon \xi \to \xi'$ and ξ' is h-oriented, then ξ is h-oriented so that $f^*\colon h(B') \to h(B)$ preserves the Euler classes.

(1.2) If ξ_1 and ξ_2 are h-oriented, then the Whitney sum $\xi_1 \oplus \xi_2$ is h-oriented so that $e(\xi_1 \oplus \xi_2) = e(\xi_1)e(\xi_2)$.

(1.3) If ξ has a non-zero cross section, then $e(\xi)=0$.

The classical Leray-Hirsch theorem on fiberings can be generalized to the multiplicative theory h, and so we have the Thom isomorphism

$$\Phi : h(B) \cong \tilde{h}(M(\xi))$$

given by $\Phi(\alpha)=\alpha\cdot t(\xi)$. As a consequence, the Gysin exact sequence

$$\cdots\to h^{i-1}(S(\xi)) \to h^{i-n}(B) \xrightarrow{\cdot e(\xi)} h^i(B) \xrightarrow{p^*} h^i(S(\xi)) \to\cdots$$

holds.

A complex vector bundle ξ is called h-orientable if the real form ξ_R is h-orientable. Let η_n denote the canonical complex line bundle over the complex n-dimensional projective space CP^n. Throughout this section the following will be assumed:

(1.4) For each n, η_n is h-oriented so that the homomorphism $h(CP^{n+1})\to h(CP^n)$ preserves the Euler classes.

It follows from this assumption that any complex line bundle ξ over a finite CW complex is h-oriented so that the homomorphism f^* : $h(B')\to h(B)$ induced by every bundle map f : $\xi\to\xi'$ preserves the Euler classes.

We can prove

(1.5) The algebra $h(CP^n)$ is a truncated polynomial algebra over $h(pt)$:

$$h(CP^n) = h(pt)[e(\eta_n)]/(e(\eta_n)^{n+1}).$$

(1.6) Put $e(\eta_m)_1=p_1^*\, e(\eta_m)$ and $e(\eta_n)_2=p_2^*\, e(\eta_n)$ for the projections p_1 : $CP^m \times CP^n\to CP^m$ and p_2 : $CP^m\times CP^n\to CP^n$. Then the isomorphism

$$h(CP^m\times CP^n) = h(pt)[e(\eta_m)_1,\ e(\eta_n)_2]/(e(\eta_m)_1^{m+1},\ e(\eta_n)_2^{n+1})$$

holds.

For a CW complex X with finite skelta, we define $h(X)$ as the inverse limit with respect to skelta :

$$h(X) = \varprojlim h(X^n).$$

Then, for the infinite dimensional projective space CP^∞, the following result is obtained from (1.5) and (1.6).

(1.7) $h(CP^\infty)$ and $h(CP^\infty\times CP^\infty)$ are rings of formal power series :

$$h(CP^\infty) = h(pt)[[x]], \qquad h(CP^\infty\times CP^\infty) = h(pt)[[x_1,\ x_2]],$$

where x, x_1 x_2 are the elements defined by $e(\eta_n)$, $e(\eta_n)_1$, $e(\eta_n)_2$ respectively.

Let η denote the canonical line bundle over CP^∞, and consider the external tensor product $\eta\hat{\otimes}\eta$ which is a complex line bundle over $CP^\infty\times CP^\infty$. Let μ : $CP^\infty\times CP^\infty\to CP^\infty$ be a classifying map for $\eta\hat{\otimes}\eta$ which is cellular, and put

$$\mu^*(x) = \sum_{i,j\geq 0} a_{ij}\, x_1^i\, x_2^j \qquad (a_{ij}\in h^{2(1-i-j)}(pt))$$

for μ^* : $h(CP^\infty)\to h(CP^\infty\times CP^\infty)$. Then we obtain easily

(1.8) For the tensor product $\xi_1\otimes\xi_2$ of any complex line bundles ξ_1 and ξ_2

over a finite CW complex,

$$e(\xi_1 \otimes \xi_2) = \sum_{i,j\geqq 0} a_{ij}\, e(\xi_1)^i\, e(\xi_2)^j$$

holds.

Consider now a power series $F(x,y)$ with coefficients in $h(pt)$, which is defined by

$$F(x,y) = \sum_{i,j\geqq 0} a_{ij}\, x^i\, y^j$$

with a_{ij} above. Then it follows that $F(x,y)$ is a formal group law over $h(pt)$, *i.e.* the identities

$$F(x, 0) = x,\ F(x, y) = F(y, x),$$
$$F(x, F(y, z)) = F(F(x, y), z)$$

hold. For each integer $i \geqq 1$, let $[i](x) \in h[[x]]$ denote the operation of "multiplication by i" for the formal group, *i.e.*

$$[1](x) = x, \qquad [i](x) = F([i-1](x), x).$$

Since the formula in (1.8) is rewritten as

$$e(\xi_1 \otimes \xi_2) = F(e(\xi_1), e(\xi_2)),$$

for the i-fold tensor product $\xi^i = \xi \otimes \cdots \otimes \xi$ we have

$$e(\xi^i) = [i](e(\xi)).$$

Given a positive integer q, let G denote a cyclic group of order q. Define a G-action on the standard $(2n+1)$-sphere $S^{2n+1} = \{(z_0, z_1, \cdots, z_n) \in \boldsymbol{C}^{n+1} \mid \sum_i |z_i|^2 = 1\}$ by

$$(z_0, \cdots, z_n)\, g_0 = (z_0 \exp 2\pi\sqrt{-1}/q, \cdots, z_n \exp 2\pi\sqrt{-1}/q),$$

where g_0 is the generator of G. This yields a principal G-bundle $\rho_n' : S^{2n+1} \to L^n(q)$ over the lens space $L^n(q)$. Let L denote a 1-dimensional complex G-module given by $c \cdot g_0 = c \exp 2\pi\sqrt{-1}/q$, and consider the associated complex line bundle $\rho_n = \rho_n' \times_G L$. For the canonical projection $\pi : L^n(q) \to CP^n$ we have $\rho_n = \pi^*(\eta_n)$, and hence $e(\rho_n)^{n+1} = 0$ holds.

Proposition 1. *Let $P(x) \in h(pt)[[x]]$. Then the element $P(e(\rho_n))$ of $h(L^n(q))$ is zero if and only if $P(x)$ is in the ideal generated by x^{n+1} and $[q](x)$.*

Proof. Consider the q-fold tensor product η_n^q of η_n. As is observed in [9], the total space $S(\eta_n^q)$ of the sphere bundle associated to η_n^q is homeomorphic with $L^n(q)$. Therefore we have the Gysin sequence

$$\cdots \to h^{i-2}(CP^n) \xrightarrow{\cdot e(\eta_n^q)} h^i(CP^n) \xrightarrow{\pi^*} h^i(L^n(q)) \to \cdots.$$

Since $e(\eta_n^q)=[q](e(\eta_n))$, the desired result follows from the above sequence and (1.5).

2. The element $s^*(\theta)$

As in § 1, let G denote a cyclic group of order q. We shall assume in the following that q is *odd*.

For any space X, let XG denote the product of q copies of X. Writing its elements as $\sum_{g\in G} x_g g$, a G-action on XG is given by

$$(\sum_{g\in G} x_g g)\cdot h = \sum_{g\in G} x_{gh^{-1}} g \qquad (h\in G).$$

We denote by ΔX the diagonal in XG.

Let Σ be a homotopy $(2n+1)$-sphere (which is a differentiable manifold), and assume that there is given a free differentiable G-action on Σ. We denote by Σ_G the orbit space.

Let M be a differentiable manifold, and consider the diagonal action on $\Sigma \times MG$ whose orbit space is denoted by $\Sigma \underset{G}{\times} MG$. $\Sigma \times \Delta M$ is an invariant submanifold of the G-manifold $\Sigma \times MG$, and its orbit space is regarded as $\Sigma_G \times \Delta M$. We denote by ν the normal bundle of $\Sigma_G \times \Delta M$ in $\Sigma \underset{G}{\times} MG$. This is a real $m(q-1)$-dimensional vector bundle.

Choose a point $y_0 \in M$, and identify Σ_G with a subspace $\Sigma_G \times y_0 G$ ($y_0 G = \sum_g y_0 g$) of $\Sigma_G \times \Delta M$.

Let $\lambda' : \Sigma \to \Sigma_G$ denote the principal G-bundle defined by the G-action on Σ, and consider the associated complex line bundle $\lambda = \lambda' \underset{G}{\times} L$.

Proposition 2. *The normal bundle ν has a complex structure for which*

$$i^*(\nu) = m(\lambda \oplus \lambda^2 \oplus \cdots \oplus \lambda^{(q-1)/2})$$

holds, where $i : \Sigma_G \to \Sigma_G \times \Delta M$ is the inclusion.

Proof. If $\nu_1 : N_1 \to \Delta M$ denote the normal G-vector bundle of ΔM in MG, then we have $\nu = id \underset{G}{\times} \nu_1 : \Sigma \underset{G}{\times} N_1 \to \Sigma_G \times \Delta M$. Therefore it suffices to prove that there exists a G-equivariant complex structure on ν_1 with the fiber over

y_0G being $m(L\oplus\cdots\oplus L^{(q-1)/2})$.

To prove this, let IG be defined by the exact sequence of real G-modules

$$0 \to \Delta \boldsymbol{R} \to \boldsymbol{R}G \to IG \to 0.$$

View this as a sequence of real G-vector bundles over a point, and identify ΔM with $M\times pt=M$ in the obvious way. Then we have the exact sequence

$$0 \to \tau M\hat{\otimes}\Delta \boldsymbol{R} \to \tau M\hat{\otimes}\boldsymbol{R}G \to \tau M\hat{\otimes}IG \to 0$$

of real G-vector bundles over M, where τM denotes the tangent bundle over M. Since $\tau(MG)=(\tau M)G$, an equivariant isomorphism

$$\beta : \tau(MG)\,|\,\Delta M \to \tau M\hat{\otimes}\boldsymbol{R}G$$

can be given by

$$\beta(\sum_g v_g g) = \sum_g v_g\otimes g \qquad (v_g\in\tau_y(M),\ y\in M).$$

Since $\sum v_g g$ is in $\tau(\Delta M)$ if and only if all v_g are equal, β maps $\tau(\Delta M)$ onto $\tau M\hat{\otimes}\Delta\boldsymbol{R}$. Thus it holds that $\nu_1\cong\tau M\hat{\otimes} IG$ as real G-vector bundles. From elementary representation theory of groups, it follows that IG is the real form of $L\oplus\cdots\oplus L^{(q-1)/2}$. This gives ν_1 its complex structure, and we get

$$\begin{aligned}(\nu_1)_{y_0} &= \tau_{y_0}M\otimes(L\oplus\cdots\oplus L^{(q-1)/2})\\ &= \boldsymbol{R}^m\otimes(L\oplus\cdots\oplus L^{(q-1)/2}) = m(L\oplus\cdots\oplus L^{(q-1)/2})\end{aligned}$$

as desired. This completes the proof.

As in § 1, let h be a given multiplicative cohomology theory. In the following we shall assume the following conditions:

(2.1) every complex vector bundle of any dimension is h-orientable.

(2.2) $h^{odd}(pt)=0$.

Assuming that M is closed, consider the normal bundle ν. Then, by Proposition 2 and (2.1), we have a Thom class $t(\nu)\in\tilde{h}^{m(q-1)}(M(\nu))$ and the corresponding Euler class $e(\nu)\in h^{m(q-1)}(\Sigma_G\times\Delta M)$ such that

$$\text{(2.3)}\qquad \begin{aligned} i^*e(\nu) &= e(m(\lambda\oplus\lambda^2\oplus\cdots\oplus\lambda^{(q-1)/2}))\\ &= (\prod_{i=1}^{(q-1)/2}[i](e(\lambda)))^m.\end{aligned}$$

As usual we shall regard the total space N of ν as a tubular neighborhood of $\Sigma_G\times\Delta M$ in $\Sigma\underset{G}{\times} MG$. Then we can identify $\tilde{h}(M(\nu))$ with $h(\Sigma\underset{G}{\times} MG,$

$\Sigma \underset{G}{\times} MG - N)$ canonically. Let

$$\theta \in h^{m(q-1)}(\Sigma \underset{G}{\times} MG)$$

be the image of the Thom class $t(\nu)$ under the homomorphism $l^* : h(\Sigma \underset{G}{\times} MG, \Sigma \underset{G}{\times} MG - N) \to h(\Sigma \underset{G}{\times} MG)$ induced by the inclusion. We have immediately

(2.4) For the homomorphism $j^* : h(\Sigma \underset{G}{\times} MG) \to h(\Sigma_G \times \Delta M)$ induced by the inclusion, $j^*(\theta) = e(\nu)$ holds.

Given a continuous map $f : \Sigma \to M$, define a continuous map $s : \Sigma_G \to \Sigma \underset{G}{\times} MG$ by

$$s(xG) = (x, \sum_g f(xg^{-1})g)G.$$

For the projection $p : \Sigma \underset{G}{\times} MG \to \Sigma_G$, $p \circ s$ is the identity.

Proposition 3. *For the homomorphism* $s^* : h(\Sigma \underset{G}{\times} MG) \to h(\Sigma_G)$ *and the homomorphism* $i^* : h(\Sigma_G \times \Delta M) \to h(\Sigma_G)$, *we have*

$$s^*(\theta) = i^*(e(\nu)).$$

Proof. It is easily seen that there exist a continuous map $f_1 : \Sigma \to M$ and an open set V of Σ satisfying the following conditions: i) f is homotopic to f_1, ii) V is homeomorphic to $\boldsymbol{R}^{2n+1}$, iii) $f_1(\Sigma - V) = y_0$, iv) $xg \notin \bar{V}$ for any $g \neq 1$ and any $x \in \bar{V}$, where $\bar{V}$ denotes the closure of V Define $s_1 : \Sigma_G \to \Sigma \underset{G}{\times} MG$ from f_1 as s was defined from f, then s and s_1 are homotopic. Let $(MG)_1$ denote the subspace of MG consisting of points with at most one coordinate $\neq y_0$ Then $(MG)_1$ is an invariant subspace of the G-space MG, and the orbit space $\Sigma \underset{G}{\times} (MG)_1$ contains $s_1(\Sigma_G)$. Since $\Sigma - V$ is contractible, there exists a homotopy $\psi_t : (\bar{V}, \partial V) \to (\Sigma, \Sigma - V)$ such that ψ_0 is the inclusion and $\psi_1(\partial V) = x_0 \in \partial V$, where $\partial V = \bar{V} - V$. Put $V_G = \pi(V)$ for the projection $\pi : \Sigma \to \Sigma_G$. Consider now the following commutative diagram:

$$\begin{array}{ccc}
\Sigma_G & \xrightarrow{\quad s_1 \quad} & \Sigma \underset{G}{\times} (MG)_1 \\
\downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_1} \\
(\Sigma_G, \Sigma_G - V_G) & \xrightarrow{\quad s_1 \quad} & (\Sigma \underset{G}{\times} (MG)_1, \Sigma_G \times y_0 G)
\end{array}$$

where j_1, j_2, are the inclusions.

We have

$$h^{m(q-1)}(\Sigma_G, \Sigma_G - V_G) = \tilde{h}^{m(q-1)}(S^{2n+1}) = h^{m(q-1)-(2n+1)}(pt) = 0$$

by (2.2). Therefore

$$s_1^* \circ i_1^* : h^{m(q-1)}(\Sigma \underset{G}{\times} (MG)_1, \Sigma_G \times y_0 G) \to h^{m(q-1)}(\Sigma_G)$$

is trivial.

Next consider the commutative diagram

$$\begin{tikzcd}
h(\Sigma \underset{G}{\times} MG) \arrow[rr, "j^*"] \arrow[dr, "i_2^*"] \arrow[dd, "s^*"'] & & h(\Sigma_G \times \Delta M) \arrow[dd, "i^*"] \\
& h(\Sigma \underset{G}{\times} (MG)_1) \arrow[dl, "s_1^*"'] \arrow[dr, "i_1^*"] & \\
h(\Sigma_G) \arrow[r, equal] & h(\Sigma_G) \arrow[u, "p^*"] \arrow[r, equal] & h(\Sigma_G \times y_0 G)
\end{tikzcd}$$

where i_1, i_2 are the inclusions. Putting $\theta' = p^* i_1^* i_2^*(\theta) - i_2^*(\theta)$, we have

$$s_1^*(\theta') = i^* i^*(\theta) - s^*(\theta) = i^*(e(\nu)) - s^*(\theta)$$

by (2.4), and $i_1^*(\theta') = 0$. Therefore θ' is in the image of $j_1^* : h^{m(q-1)}(\Sigma \underset{G}{\times} (MG)_1, \Sigma_G \times y_0 G) \to h^{m(q-1)}(\Sigma \underset{G}{\times} (MG)_1)$, and hence $s_1^*(\theta') = 0$ by the fact proved above. Thus we have $i^*(e(\nu)) = s^*(\theta)$.

3. Generalization of Borsuk-Ulam theorem

Let Σ be as in §2, and let $f : \Sigma \to M$ be a continuous map to a differentiable m-manifold. Put

$$A(f) = \{x \in \Sigma \mid f(x) = f(xg) \text{ for any } g \in G\}.$$

In this section we shall consider the covering dimension of $A(f)$.

For the image $A(f)_G = \pi(A(f))$, we have $\dim A(f) = \dim A(f)_G$.

Proposition 4. *Assume that M is closed. Then $\dim A(f) < 2d$ implies*

$$e(d\lambda)s^*(\theta) = 0.$$

Proof. Since $\dim A(f)_G \leqq 2d-1$, it follows that $d\lambda$ has a non-zero cross section over $A(f)_G$ (see [5], Lemma 2). By standard facts on extension of cross section, this cross section extends to a non-zero cross section over the closure $\bar{W}$ of some neighborhood W of $A(f)_G$ in Σ_G. Here we may assume that $\bar{W}$ is a finite CW complex, and that $s(\Sigma_G - W) \subset \Sigma \underset{G}{\times} MG - N$ by taking N small. We have then $e(d\lambda \mid \bar{W}) = 0$, and so $e(d\lambda)$ is in the image of $l_1^* : h(\Sigma_G, \bar{W}) \to h(\Sigma_G)$ induced by the inclusion.

On the other hand, it follows from the commutative diagram

$$\begin{array}{ccc} h(\Sigma \underset{G}{\times} MG, \Sigma \underset{G}{\times} MG - N) & \xrightarrow{l^*} & h(\Sigma \underset{G}{\times} MG) \\ \downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle s^*} \\ h(\Sigma_G, \Sigma_G - W) & \xrightarrow{l_2^*} & h(\Sigma_G) \end{array}$$

(l, l_2 : inclusions) that $s^*(\theta)$ is in the image of l_2^*.

Therefore $e(d\lambda)\, s^*(\theta)$ is in the image of the homomorphism $h(\Sigma_G, \bar{W} \cup (\Sigma_G - W)) = h(\Sigma_G, \Sigma_G) \to h(\Sigma_G)$, and hence we have the desired result.

We shall now prove the main theorem.

Theorem 1. *Let G be a cyclic group of odd order q, and Σ be a homotopy $(2n+1)$-sphere on which a free differentiable G-action is given. Let M be a differentiable m-manifold. Assume that there exists a continuous map $f : \Sigma \to M$ with dim $A(f) < 2d$. Then, for any multiplicative cohomology theory h defined on the category of finite CW pairs and satisfying the conditions* (2.1), (2.2), *it holds that*

$$x^d \Big(\prod_{i=1}^{(q-1)/2} [i](x) \Big)^m \in h(pt)[[x]]$$

is contained in the ideal generated by x^{n+1} and $[q](x)$.

Proof. Recall that any differentiable m-manifold is regarded as an increasing union of compact differentiable m-manifold, and that any differentiable m-manifold with boundary is contained in a differentiable m-manifold without boundary. Since Σ is connected and compact, it follows from these facts that we may assume M to be closed without loss of generality.

Then, in virtue of (2.3), Propositions 3 ane 4, we have

$$\begin{aligned} & e(\lambda)^d \Big(\prod_{i=1}^{(q-1)/2} [i](e(\lambda)) \Big)^m \\ & = e(d\lambda) \cdot i^* e(\nu) = e(d\lambda) \cdot s^*(\theta) = 0. \end{aligned}$$

Since ρ_n' is a principal G-bundle whose base space is $(2n+1)$-dimensional CW complex, and since λ' is a $(2n+1)$-universal principal G-bundle, there is a bundle map of ρ_n to λ. Hence the last equation implies

$$e(\rho_n)^d \Big(\prod_{i=1}^{(q-1)/2} [i](e(\rho_n)) \Big)^m = 0.$$

From this and Proposition 1 we have the desired result.

As typical examples of the multiplicative cohomology theory satisfying the conditions in Theorem 1, we have the classical integral cohomology theory $H^*(\ ;\boldsymbol{Z})$, the Grothendieck-Atiyah-Hirzebruch periodic cohomology theory $K^*(\ )$ of K-theory, and the complex cobordism theory $U^*(\ )$ obtained from the Milnor spectrum MU (see [2]).

As is well known, $H^i(pt;\boldsymbol{Z})=\boldsymbol{Z}$ $(i=0)$, $=0$ $(i\neq 0)$ and the formal group law for $H^*(\ ;\boldsymbol{Z})$ is given by $F(x, y)=x+y$. Hence the conclusion in Theorem 1 for $h=H^*(\ ;\boldsymbol{Z})$ is stated that

$$\left(\frac{q-1}{2}!\right)^m x^{d+m(q-1)/2}\in\boldsymbol{Z}[x]$$

is contained in the ideal generated by x^{n+1} and qx. From this we obtain the following result.

(3.1) If q is an odd prime, for any continuous map $f:\Sigma\to M$ we have $\dim A(f)\geqq 2n-m(q-1)$.

REMARK. The conclusion in (3.1) is strengthened to $\dim A(f)\geqq 2n+1-m(q-1)$ (see [4], [6]).

For $K^*(\ )$ it is known that $K^{even}(pt)=\boldsymbol{Z}$, $K^{odd}(pt)=0$ and the formal group law is given by $F(x, y)=x+y+xy$ (see[1]). Therefore the conclusion in Theorem 1 for $h=K^*(\ )$ is stated that

$$x^d\Big(\prod_{i=1}^{(q-1)/2}((x+1)^i-1)\Big)^m\in\boldsymbol{Z}[x]$$

is contained in the ideal generated by x^{n+1} and $(x+1)^q-1$. Putting $y=x+1$ this is restated that

$$(y-1)^d\Big(\prod_{i=1}^{(q-1)/2}(y^i-1)\Big)^m\in\boldsymbol{Z}[y]$$

is contained in the ideal generated by $(y-1)^{n+1}$ and y^q-1. If q is an odd prime power p^a, it can be proved by making use of elementary algebraic number theory that the above statement is equivalent to

$$d\geqq n+p^{a-1}-am(p^a-p^{a-1})/2$$

(see [5], p. 453). Thus theorem 1 implies the following theorem containing (3.1) and being a generalization of the main result in [5].

Theorem 2. *If q is an odd prime power p^a, for any continuous map $f:\Sigma\to M$ we have*

$$\begin{aligned}\dim A(f)\geqq\ &2n+1-(p^a-1)m\\ &-[m(a-1)p^a-(ma+2)p^{a-1}+m+3].\end{aligned}$$

For $U^*(\ \ )$, it is known that $U^*(pt)$ is a polynomial ring over $\boldsymbol{Z}$ with one generator of degree $-2i$ for each positive integer i, and that the formal group law for $U^*(\ \ )$ is given by

$$F(x, y) = g^{-1}(g(x)+g(y))$$

with $g(x)=\sum_{i\geqq 0}\frac{[CP^i]}{i+1}x^{i+1}\in U^*(pt)[[x]]\otimes \boldsymbol{Q}$, where $\boldsymbol{Q}$ is the ring of rational numbers (see [1], [7]). However we can not deduce numerical conditions equivalent to the conclusion in Theorem 1 for $h=U^*(\ \ )$.

Appendix

In this appendix we shall show a generalization of a result due to Vick [10].

For any positive integer r, let $T_r : S^{2n+1}\to S^{2n+1}$ denote the fixed point free transformation of period r given by

$$T_r(z_1,\cdots, z_{n+1})=(z_1 \exp 2\pi\sqrt{-1}/r,\cdots,z_n \exp 2\pi\sqrt{-1}/r).$$

Then a fixed point free transformation $\bar{T}_p : L^n(q)\to L^n(q)$ of period p on the lens space $L^n(q)$ is induced by $T_{pq} : S^{2n+1}\to S^{2n+1}$.

Theorem 3. *Suppose that there exists an equivariant map f of $(L^n(q), \bar{T}_p)$ to (S^{2m+1}, T_p). Then, for any multiplicative cohomology theory h defined on the category of finite CW pairs and satisfying* (1.4), *it holds that* $([q](x))^{m+1}\in h(pt)[[x]]$ *is contained in the ideal generated by x^{n+1} and $[pq](x)$.*

Proof. For a multiple pq of q, let $\rho'(q, pq)$ denote the principal $\boldsymbol{Z}_p$-bundle $L^n(q)\to L^n(pq)$ defined the canonical projection. Corresponding to the standard 1-dimensional complex representation of $\boldsymbol{Z}_p$, we have the associated complex line bundle $\rho_n(q, pq)$ on $L^n(pq)$. As is observed in [8], it holds that

$$\rho_n(q, pq)\cong\rho_n(1, pq)\otimes\cdots\otimes\rho_n(1, pq) \qquad (q\text{-times}).$$

Therefore, if there exists an equivariant map $f : (L^n(q), \bar{T}_p)\to(S^{2m+1}, T_p)$, then it holds that

$$\bar{f}^*\rho_m(1, p)\cong\rho_n(1, pq)\otimes\cdots\otimes\rho_n(1, pq) \qquad (q\text{-times})$$

for the map $\bar{f} : L^n(pq)\to L^m(p)$ induced by f.

$$\begin{tikzcd} S^{2n+1} \arrow[r, "\rho'_n(1{,}q)"] \arrow[dr, "\rho'_n(1{,}\, pq)"'] & L^n(q) \arrow[r, "f"] \arrow[d, "\rho'_n(q{,}pq)"] & S^{2m+1} \arrow[d, "\rho'_m(1{,}p)"] \\ & L^n(pq) \arrow[r, "\bar{f}"] & L^m(p) \end{tikzcd}$$

Therefore we have

$$\bar{f}^*e(\rho_m(1, p)) = [q](e(\rho_n(1, pq)))$$

in $h(L^n(pq))$. Since $e(\rho_m(1, p))^{m+1}=0$ it holds that

$$([q](e(\rho_n(1, pq))))^{m+1} = 0$$

in $h(L^n(pq))$. This and Proposition 1 prove the desired result.

The conclusion of Theorem 3 applied to $h=K^*(\ )$ is stated that $((x+1)^q-1)^{m+1}\in \boldsymbol{Z}[x]$ is contained in the ideal generated by x^{n+1} and $(x+1)^{pq}-1$. Therefore, the argument similar to the proof of Lemma 1 in [5] proves the following

Theorem 4. *Let p be a prime, and suppose that there exists an equivariant map of $(L^n(q), \bar{T}_p)$ to (S^{2m+1}, T_p). Then we have*

$$p^a m \geqq n,$$

where $q=p^a r$, $(p, r)=1$.

REMARK 1. This generalizes the result due to Vick [10].

REMARK 2. Shibata [8] proves this result by applying Theorem 3 to $h=U^*(\ )$.

(added in proof) Since the formal group law for the complex cobordism theory is universal (see [1], [7]), we have the following corollary of Theorem 1 : *For any formal group law over a commutative ring R with unit, it holds that*

$$\dim A(f)<2d \Rightarrow$$

$$x^d\Big(\prod_{i=1}^{(q-1)/2}[i](x)\Big)^m\subset(x^{n+1}, [q](x)) \textit{ in } R[[x]]\,.$$

Similar for Theorem 3. This fact was pointed out by J. Morava.

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